

# On the Interpretability and Compact Representation of Linguistic Fuzzy Systems

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**Abstract.** Finding the compromise between computational complexity and adaptation potential of linguistic fuzzy systems is important in several fields of application of fuzzy systems including fuzzy modeling and control. This paper considers the role of popular s- and t-norms in fuzzy inference function in this aspect and presents some recently acquired results. First, it is shown that with simultaneous application of product implication and sum aggregation, it is sufficient to consider symmetrical triangular output MFs, which also makes output-side transparency a default property. Secondly, analytical inference function for linguistic fuzzy systems utilizing minimum implication is derived for several cases of output MFs. Finally, some aspects of maximum aggregation are observed. It appears that computational complexity and transparency are somewhat compatible and the compromise is thus attainable as adaptability potential is difficult to realize when inference algorithm becomes too complex because of lack of capable methods and inherent mathematical limitations.

**Keywords.** Fuzzy logic and systems, fuzzy modeling, inference algorithm, transparency to interpretation.

## 1 Introduction

One problem that plagues linguistic fuzzy (Mamdani) systems is that generally the mathematical description of the relationship between the system input and output variables is non-analytical and/or very complex. This has the following results (a) computation of the relationship between the system variables becomes computationally expensive; (b) it is difficult or impossible to apply the tools of mathematical analysis, e.g. control loop stability routines in fuzzy control; (c) it prevents us from representing fuzzy systems as neural networks (in order to apply the powerful learning techniques for their identification from data) or makes it very difficult.

In recent years therefore, the focus has shifted to what are called 0<sup>th</sup> or 1<sup>st</sup> order Takagi-Sugeno (TS) systems [1] or a “simplified” method of reasoning [2] (that reduces a linguistic fuzzy system into a *de facto* 0<sup>th</sup> order TS system). The implications are: interpretation of fuzzy rules becomes inconvenient due to half-linguistic/half-linguistic nature fuzzy rules (1st order TS systems); limited interpolation properties (0<sup>th</sup> order TS systems) that inevitably leads to explosion of rules in more complex research problems.

It is, however, not fully justified to follow this trend because it is possible to give analytical and compact representation for linguistic fuzzy systems utilizing wide choice of output membership functions and even non-analytical t-norms, i.e. minimum premise conjunction and implication as this paper attempts to demonstrate.

## 2 System Definition

We consider multi-input/single-output linguistic fuzzy systems, based on the following rule format.

$$\text{IF } x_1 \text{ is } A_{1r} \text{ AND } x_2 \text{ is } A_{2r} \text{ AND } \dots \text{ AND } x_N \text{ is } A_{Nr} \text{ THEN } y \text{ is } B_r, \quad (1)$$

where  $A_{ir}$  and  $B_r$  denote the linguistic labels of  $i^{\text{th}}$  input variable  $x_i$  and output variable  $y$  ( $i = 1 \dots N$ ), respectively, associated with the  $r^{\text{th}}$  rule ( $r = 1 \dots R$ ).

Besides giving the linguistic description of system behavior in the form of (1), fuzzy systems also perform numerical evaluation of the functional relationship between system variables using the inference algorithm that is described below.

The algorithm consists of four steps: fuzzification, premise conjunction, implication and aggregation. The result, fuzzy output ( $F(y)$ ) of the system is then defuzzified using mostly and preferably center-of-gravity defuzzification (2).

$$y = Y_{cog}(F(y)) = \frac{\int_{y_{min}}^{y_{max}} F(y)y dy}{\int_{y_{min}}^{y_{max}} F(y) dy}. \quad (2)$$

As mentioned,  $F(y)$  itself is obtained by aggregating individual rule outputs  $F_r(y)$  using a s-norm operator (denoted by  $\cup$  in (3))

$$F(y) = \bigcup_{r=1}^R F_r(y). \quad (3)$$

$F_r(y)$  in turn is computed by performing implication (a t-norm operator denoted by  $\cap$ ) on rule activation degree  $\tau_r$  and output membership function (MF) associated with the given rule

$$F_r(y) = \tau_r \cap \gamma_r(y), \quad (4)$$

where  $\tau_r$  is obtained by applying premise conjunction to input MFs associated with the  $r^{\text{th}}$  rule.

$$\tau_r = \bigcap_{i=1}^N \mu_{ir}(x_i) \quad (5)$$

Note that MFs  $\mu_{ir}$  in (5) and in  $\gamma_r$  in (4) have one-to-one correspondence to linguistic labels  $A_{ir}$  and  $B_r$  in (1), respectively, and have therefore crucial importance in interpreting the rules and system behavior.

Collecting (3-5) into (2) the whole body of the algorithm is expressed by

$$y = \frac{\int_{y_{min}}^{y_{max}} \bigcup_{r=1}^R \left( \bigcap_{i=1}^N \mu_{ir}(x_i) \cap \gamma_r(y) \right) y dy}{\int_{y_{min}}^{y_{max}} \bigcup_{r=1}^R \left( \bigcap_{i=1}^N \mu_{ir}(x_i) \cap \gamma_r(y) \right) dy}. \quad (6)$$

### 3 Transparency constraints

Transparency of fuzzy systems is one of the very essential presumptions for their interpretability [3]. The mission of transparency constraints is to ensure conformity between the linguistic and inference layer of the system (1,6). In short, transparency can be interpreted as a measure of validity of interpretation. It is important to acknowledge that it is not a default property of fuzzy systems. In case of linguistic fuzzy systems, the definition of transparency, however, is quite simple and of binary nature. We need to observe that

$$\forall r, \tau_r = 1: y = \text{core}(\gamma_r). \quad (7)$$

The issue is covered in more detail in [3]. For the needs of current paper we note that (7) is satisfied if both.

$$\forall x_i \in X_i : \sum_{s=1}^{S_i} \mu_i^s(x_i) \leq 1, \quad (8)$$

and

$$Y_{cog}(\gamma_r(y)) = \frac{\int_{y_{min}}^{y_{max}} y \gamma_r(y) dy}{\int_{y_{min}}^{y_{max}} \gamma_r(y) dy} = \text{core}(\gamma_r(y)) \quad (9)$$

are valid. Note that (8) is based on the assumption that we employ parameterized piecewise linear (such as triangular or trapezoid) MFs.

Note that the s- and t-norms operators commonly used (minimum/algebraic product, maximum/sum) do not affect system transparency, rather determine rule interpolation in regions where  $\tau_r < 1$  [4] and highly influence the representation of inference function which is observed in the following sections.

### 4 Inference with product implication and sum aggregation

Assuming product implication and sum aggregation (for reasons that become clearer below), discrete “by-the-book” implementation of inference function follows

$$\begin{aligned}
 Y_{cog}(F(y)) &= \frac{\left( \sum_{r=1}^R \tau_r \cdot \Gamma_r \right) Y^T}{\left( \sum_{r=1}^R \tau_r \cdot \Gamma_r \right) \mathbf{1}} = \\
 &= \frac{\sum_{q=1}^Q \sum_{r=1}^R \tau_r \gamma_r(y_q) y_q}{\sum_{q=1}^Q \sum_{r=1}^R \tau_r \gamma_r(y_q)} = \frac{\sum_{r=1}^R \tau_r \sum_{q=1}^Q \gamma_r(y_q) y_q}{\sum_{r=1}^R \tau_r \sum_{q=1}^Q \gamma_r(y_q)}, \quad (10)
 \end{aligned}$$

where  $\Gamma_r = [\gamma_r(y_1) \gamma_r(y_2) \dots \gamma_r(y_q) \dots \gamma_r(y_Q)]$ ,  $Y = [y_1 y_2 \dots y_q \dots y_Q]$  and  $\mathbf{1}$  is unitary column vector of  $Q$  elements. Note that sum-aggregation enables us to regroup the additives in (10). Now, assuming that the discretization interval of  $y$  is approaching zero ( $Q \rightarrow \infty$ ), (7) can be rewritten as

$$y = \frac{\sum_{r=1}^R \tau_r \int_{y_{\min}}^{y_{\max}} \gamma_r(y) y dy}{\sum_{r=1}^R \tau_r \int_{y_{\min}}^{y_{\max}} \gamma_r(y) dy} = \frac{\sum_{r=1}^R \tau_r C_r S_r}{\sum_{r=1}^R \tau_r S_r}, \quad (11)$$

where  $S_r$  is the area covered by  $\gamma$

$$S_r = \int_{y_{\min}}^{y_{\max}} \gamma_r(y) dy \quad (12)$$

and  $C_r$  is its center of gravity.

$$C_r = \int_{y_{\min}}^{y_{\max}} \gamma_r(y) y dy / S_r \quad (13)$$

Thus, according to (11), application of different types of output MFs reduces to finding the values of  $S_r$  and  $C_r$  [5]. With some MF types integrals used for the computation of  $C_r$  and  $S_r$  can be indefinite (improper) and/or non-analytic (the Gaussian one is a good example) therefore the integration must be performed numerically. This, however, is not really necessary because (11) implies that on given assumptions (product implication, sum aggregation, center-of-gravity defuzzification) **any output MF can be replaced by the equivalent expression of the**

**simplest one – symmetrical triangular MF** (14) - which is well justified choice also for the second reason, expressed by (9) – symmetry of output MFs is vital for system transparency.

$$\gamma_r = \begin{cases} \frac{2y - 2b_r + s_r}{s_r}, & b_r - s_r/2 < y < b_r \\ 0, & b_r + s_r/2 \leq y \leq b_r - s_r/2 \\ \frac{2b_r - 2y + s_r}{s_r}, & b_r < y < b_r + s_r/2 \end{cases} \quad (14)$$

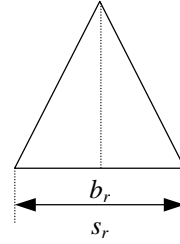


Fig. 1. Symmetrical triangular membership function

By putting respective expressions  $C_r = b_r$ ,  $S_r = s_r/2$  into (11), we obtain a very compact and convenient form of (6)

$$y = \frac{\sum_{r=1}^R \tau_r b_r s_r}{\sum_{r=1}^R \tau_r s_r} \quad (15)$$

It is interesting to note that principally similar inference function has been obtained by Kosko [6], which however, differs in interpretation ( $s_r$  are considered as rule weights).

Note also that if  $\forall r, s_r = \xi$ , meaning that all supports  $s_r$  are equal, (15) reduces to the well-known inference function of 0th order TS systems.

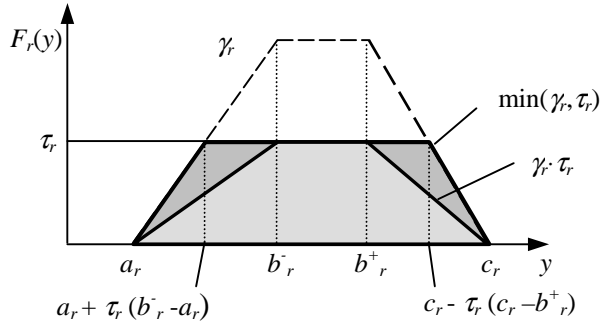
#### 4 Non-analytic t-norms

It appears that minimum premise conjunction, which is the first t-norm in the inference function used to

compute  $\tau_r$  (5) does not limit analytical optimization of fuzzy systems. First, it has been shown that fuzzy systems with minimum premise conjunction can be optimized using gradient descent method [7]. Secondly, output parameters of TS systems can be identified with least squares method [1] regardless of premise conjunction operator type. Combination of the above methods - ANFIS [8] - is one of the most effective methods in analytical optimization that in present case can then be easily realized. Obviously, the selection of premise conjunction operator does matter only in the case of multiple inputs where additional flexibility due to minimum operator can be utilized. Note that

$$\prod_{i=1}^N \mu_{i_r}(x_i) \leq \min(\mu_{1_r}(x_1), \dots, \mu_{N_r}(x_N)), \quad (16)$$

thus product operator generally provides less varying, smoother interpolation.



**Fig. 2.** Fuzzy rule output with different implication operators

It was shown in the previous section that if we choose product implication, sum aggregation and center-of-gravity defuzzification method then there is no real necessity to consider other types of MFs than symmetrical triangles (even if there is, accurate conversion to symmetrical triangles can be made any time using (12-13). The situation is different if minimum implication is applied instead of product – the difference in fuzzy rule output  $F_r(y)$  is shown in Fig. 2. Note that  $\gamma_r$  is defined as symmetrical trapezoid.

The resulting trapezoid implied by  $\min(\tau_r, \gamma_r)$  is defined by (17)

$$F_r(y) = \begin{cases} \gamma_r, & \text{if } a_r < y < a_r + \tau_r(b_r^- - a_r) \\ \tau_r, & \text{if } a_r + \tau_r(b_r^- - a_r) < y < c_r - \tau_r(c_r - b_r^+) \\ \gamma_r, & \text{if } c_r - \tau_r(c_r - b_r^+) < y < c_r \\ 0, & \text{otherwise} \end{cases} \quad (17)$$

whereas defuzzified output is computed by

$$y = \frac{\sum_{r=1}^R \int_{y_{\min}}^{y_{\max}} F_r(y) y dy}{\sum_{r=1}^R \int_{y_{\min}}^{y_{\max}} F_r(y) dy} = \frac{\sum_{r=1}^R C_r(F_r(y)) S_r(F_r(y))}{\sum_{r=1}^R S_r(F_r(y))} \quad (18)$$

Analytical derivation of inference function becomes more difficult ( $\tau_r$  cannot be separated). If we are using symmetrical trapezoids  $\gamma_r$  (as in Fig.2)

$$C(F_r(y)) = \frac{b_r^- + b_r^+}{2}, \quad (19)$$

$$S(F_r(y)) = \tau_r \left( s_r - \tau_r \frac{b_r^- - b_r^+ + s_r}{2} \right)$$

With symmetrical triangles  $b_r^- = b_r^+ = b_r$ , thus

$$C(F_r(y)) = b_r, \quad S(F_r(y)) = \tau_r s_r \left( 1 - \frac{\tau_r}{2} \right) \quad (20)$$

As we see minimum implication the type of output MFs does matter, offering extra degrees of freedom, and the expressions (19-20) give analytically computable inference functions. In simplest case (20), the expression (18) would reduce to

$$y = \frac{\sum_{r=1}^R \tau_r b_r s_r (1 - \tau_r / 2)}{\sum_{r=1}^R \tau_r s_r (1 - \tau_r / 2)} \quad (21)$$

### 5 Additional Comments

We have discarded maximum aggregation for the sake of compactness and analyticity. In present section we discuss the cost of the exclusion this particular s-norm as aggregation operator.

Admittedly, maximum aggregation offers extra degrees of freedom when compared to sum aggregation. With sum aggregation (and assuming product implication), it is only the relative size of the supports  $s_r$  of contributing output MFs that has significance as the denominator and nominator of (9) can be multiplied with an arbitrary constant, which has therefore no effect on inference function. The difference with maximum aggregation is illustrated by a simple example where we consider a single-input/single-output two-rule fuzzy system.

$$\begin{aligned} &\text{IF } x \text{ is } A_r \text{ THEN } y \text{ is } B_r \\ &\text{IF } x \text{ is } A_{r+1} \text{ THEN } y \text{ is } B_{r+1} \end{aligned}$$

Triangular input MFs corresponding to  $A_r$  and  $A_{r+1}$  that have 50% overlap (Ruspini-style partition) with their modal points fixed at  $a_r$  and  $a_{r+1}$ , respectively. At the same time we vary the overlap of contributing output MFs  $\gamma_r$  and  $\gamma_{r+1}$  (Fig. 4), resulting in four different input-output relations (Fig. 3). Note that the ratio  $s_r/s_{r+1}$  is the same in all cases, meaning that with sum aggregation (and product implication) all considered fuzzy systems produce the same input-output relationship (case a, Fig. 3).

If output MFs under question do not overlap (case a, Fig. 3) maximum and sum aggregation are equivalent. This is, however, a rare case in fuzzy system engineering, because, the larger the supports  $s_r$  of simultaneously fired output MFs, the greater the probability of overlap, besides, in linguistic modeling output MFs are almost always shared (can be associated with several rules simultaneously) which further increases the probability of overlap.

The higher the overlap degree, the larger the deviation from linear interpolation characteristic to 0<sup>th</sup> order TS systems (dashed line, Fig. 3). Another interesting feature of maximum aggregation is so-called “suffocation” effect that occurs if one MF is

sufficiently “swallowed” by another and that can be recognized in flat intervals in input-output relationship (most extremely in case d, Fig. 3). This is in many cases undesirable and could be achieved by other means (e.g. by manipulating the overlap of input MFs or by using additional rules with the same output value). In order to provide smooth interpolation between the rules, overlapping degree must thus be kept in the reasonable limits that requires constrained adaptation. Furthermore, and perhaps most importantly, with maximum aggregation it is the overlap rather than the ratio of supports of contributing output MFs that has the impact on the inference function making the systems rather unintuitive from interpretational viewpoint.

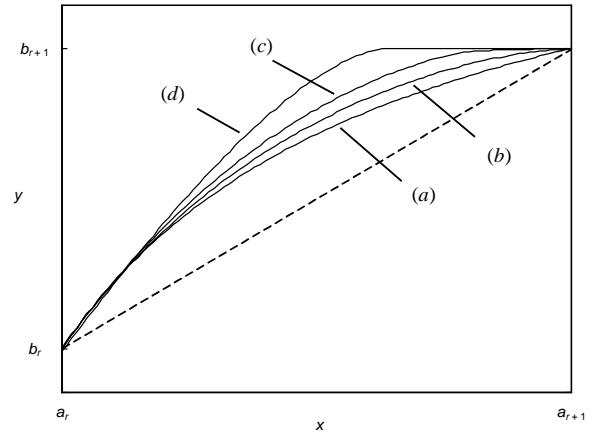


Fig. 3. Input-output relationship

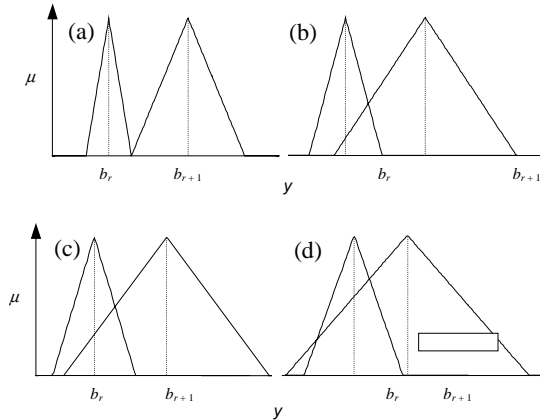


Fig. 4. Contributing output MFs of four considered fuzzy systems

## 6 Conclusions

Our study has dealt with the problem of lack of analyticity in the description of linguistic fuzzy systems and pointed out some possibilities of compromise. Use of linguistic fuzzy systems can be extended to applications requiring higher detailization level if one uses the definition (8). Note that with product implication it is only necessary consider symmetrical triangle-type output MFs. If the added degree of potential accuracy is not sufficient, one can consider the application of minimum implication and the resulting system definition (). Besides mathematical analysis of fuzzy systems such compact definitions can be extremely useful in real time (e.g. control) applications where computation time is essential to the success of the application. From this aspect, maximum aggregation that would offer even more flexibility, fuzzy systems cannot be expressed analytically and some introduced undesirable interpolation properties (section 5) render it useless also from interpretation viewpoint.

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